

# GRAVITATIONAL STABILITY OF FINITE MASSIVE BODIES

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**ABSTRACT.** Jeans instability of finite massive bodies at hydrostatic equilibrium is studied. Differential equation governing the evolution of infinitesimal disturbances is derived. We take into account radial inhomogeneity of mass density and other fluid parameters at the equilibrium state. Dispersion relation and a simple analytical formula, generalizing the Jeans criterion of instability, are derived.

## 1. INTRODUCTION

It is well known (See, for instance [1] - [6]) that the Jeans' instability forms the basis of our understanding of gravitational condensation. In particular, Jeans' mass criterion is invoked in astrophysical theories of the formation of stars, gaseous clouds, etc. Usually gravitational instability is analysed in terms of the Jeans' wavelength [3]

$$\lambda_J = \sqrt{\frac{\pi c_s^2}{G \rho_0}}, \quad (1.1)$$

or, equivalently, in terms of Jeans' mass  $M_J \sim \rho_0 \lambda_J^3$ . In this formula  $G$  is the gravitational constant,  $\rho_0$  is the unperturbed mass density and  $c_s$  is the adiabatic sound speed. As is now widely known, perturbations in homogeneous fluid with mass greater than a critical value  $M_J$  may grow producing gravitationally bounded structures. In the process of their evolution this structures can achieve states of hydrodynamic equilibrium like stars polytropes or gas clouds when pressure gradient equals gravitational force.

In this paper we investigate hydrodynamic equilibrium and stability of finite self-gravitating fluid mass with inhomogeneous distribution of mass density, pressure and temperature along the radius. In linear approach we get Schrödinger-like [8] equation with eigenvalues and eigenfunctions give us increments and profiles of disturbances.

## 2. BASIC FORMALISM

Consider spherically symmetric fluid body with radius  $R$  and mass  $M$ . We assume that the system is non-rotating and non-expanding. It can be star, gaseous cloud, etc. The

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evolution of a self-gravitating fluid is described by the conservation equations for mass, momentum and specific entropy, coupled with the Poisson equation

$$\frac{\partial \rho}{\partial t} + \nabla \rho \mathbf{v} = 0 \quad (2.1)$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \nabla) \mathbf{v} = -\nabla p - \rho \nabla \varphi \quad (2.2)$$

$$\frac{\partial s}{\partial t} + (\mathbf{v} \nabla) s = 0 \quad (2.3)$$

$$\nabla^2 \varphi = 4\pi G \rho, \quad (2.4)$$

where  $\mathbf{v}$  is the velocity,  $p$  is the pressure,  $\rho$  is the mass density,  $s$  is the specific entropy and  $\varphi$  is the gravitational potential. The linearization procedure has clearly invoked that local state variables deviate from their equilibrium values through linear fluctuation, namely

$$\begin{aligned} p &= p_0(r) + p_1(t, \mathbf{r}), & \rho &= \rho_0(r) + \rho_1(t, \mathbf{r}), \\ s &= s_0(r) + s_1(t, \mathbf{r}), & \varphi &= \varphi_0(r) + \varphi_1(t, \mathbf{r}). \end{aligned} \quad (2.5)$$

Velocity  $\mathbf{v}$  itself is infinitesimal. Substitution of (2.5) in (2.1) - (2.4) constitutes equations for equilibrium state ([1])

$$-\frac{1}{\rho_0(r)} \nabla p_0(r) - \nabla \varphi_0(r) = 0 \quad (2.6)$$

$$\nabla^2 \varphi_0(r) = 4\pi G \rho_0(r) \quad (2.7)$$

and for perturbed parameters

$$\frac{\partial \rho_1}{\partial t} + \nabla \rho_0 \mathbf{v} = 0 \quad (2.8)$$

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p_1 - \rho_1 \nabla \varphi_0 - \rho_0 \nabla \varphi_1 \quad (2.9)$$

$$\frac{\partial s_1}{\partial t} + \mathbf{v} \nabla s_0 = 0 \quad (2.10)$$

$$\nabla^2 \varphi_1 = 4\pi G \rho_1. \quad (2.11)$$

These equations must be coupled with the proper equation of state. To simplify the analysis and exclude buoyancy forces ([9],[10]) we take it to be adiabatic

$$p \sim \rho^\gamma, \quad (2.12)$$

where  $\gamma$  is the adiabatic exponent. Thus we get no entropy disturbances

$$s(t, \mathbf{r}) = s_0(r) = \text{const} \quad (2.13)$$

and pressure and mass density disturbances are bound together with adiabatic equation

$$p_1 = c_s^2 \rho_1. \quad (2.14)$$

Equations (2.6), (2.7) and (2.12) for equilibrium state are resolved with well known Emden functions ([1]) for polytropic model. For further references we summarize here basic results of polytropic theory of ideal gas when polytrope exponent  $n = 1/(\gamma - 1)$ . In this case radial dependancy of equilibrium parameters is

$$\begin{aligned} p_0(r) &= p(0)\Theta_n^{n+1}(\xi) \\ \rho_0(r) &= \rho(0)\Theta_n^n(\xi) \\ c_s^2(r) &= c_s^2(0)\Theta_n(\xi), \end{aligned} \quad (2.15)$$

where  $\xi = r/R$  and  $\Theta_n$  are the non-dimensional radius and temperature respectively. To satisfy equilibrium equations (2.6), (2.7) and boundary conditions  $\Theta(0) = 1$ ,  $\Theta(1) = 0$  parameter

$$\xi_n^2 = \frac{4\pi G \rho(0) R^2}{n c_s^2(0)} \quad (2.16)$$

must have unique value for each  $n$ . For example for  $n = 3/2$  it equals to  $\xi_{3/2} = 3.65$  and for  $n = 5/2$   $\xi_{5/2} = 5.36$  ([1]).

Equations (2.8), (2.9), (2.11) and (2.14) with proper boundary conditions forms the basis of our treatment of stability.

Boundary conditions we discuss later. But now we transform perturbed equations into the more convenient form. Substituting (2.11) into (2.8) gives us

$$\nabla \left( \frac{1}{4\pi G} \nabla \frac{\partial \varphi_1}{\partial t} + \rho_0 \mathbf{v} \right) = 0. \quad (2.17)$$

Taking into account that  $\text{div rot } \mathbf{\Psi} \equiv 0$  we get from (2.17)

$$\rho_0 \mathbf{v} = -\frac{1}{4\pi G} \nabla \frac{\partial \varphi_1}{\partial t} + \text{rot } \mathbf{\Psi} = 0, \quad (2.18)$$

where  $\mathbf{\Psi}$  stands for vector-potential of the flow  $\rho_0 \mathbf{v}$ . As we can see from (2.18), vector-potential  $\mathbf{\Psi}$  represents the “axial part” of disturbances and is not directly bound with gravitational potential. So we assume it to be zero. Next, inserting (2.18) with  $\mathbf{\Psi} = 0$  and (2.14) into (2.9), we obtain

$$-\frac{1}{4\pi G} \nabla \frac{\partial^2 \varphi_1}{\partial t^2} + \rho_0 \nabla \varphi_1 = -\nabla c_s^2 \rho_1 - \rho_1 \nabla \varphi_0. \quad (2.19)$$

A harmonic time dependence  $\sim \exp(-i\omega t)$  of the perturbations can now be assumed in terms of constant complex frequency  $\omega$  so that equation (2.19) after simple transformations becomes

$$\nabla\varphi_1 = -4\pi G \frac{\nabla c_s^2 \rho_1 + \rho_1 \nabla\varphi_0}{\omega^2 + 4\pi G \rho_0}. \quad (2.20)$$

Taking divergence of equation (2.20) and using (2.11) we finally get equation for mass density disturbance  $\rho_1$  only

$$\nabla \left( \frac{\nabla c_s^2 \rho_1 + \rho_1 \nabla\varphi_0}{\omega^2 + 4\pi G \rho_0} \right) + \rho_1 = 0. \quad (2.21)$$

Before discussing equation (2.21) we make the last simplification by introducing auxillary function

$$\chi(r) = \int_0^r \frac{dr}{c_s^2(r)} \frac{d}{dr} (\varphi_0(r) + c_s^2(r)) \quad (2.22)$$

and replace  $\rho_1$  with new function  $u$

$$\rho_1 = \exp(-\chi)u. \quad (2.23)$$

It yields final equation for unknown function  $u$

$$\exp(\chi) \nabla \left( \frac{\exp(-\chi) c_s^2}{\omega^2 + 4\pi G \rho_0} \nabla u \right) + u = 0. \quad (2.24)$$

Equation (2.24) represents a modification of Sturm-Liouville eigenvalue problem and, together with suitable boundary conditions, provides the eigenvalues and eigenfunctions for the perturbations. Boundary conditions are summarized as follows.

On the free moving surface of the body pressure must be equal to zero

$$p_1(t, r = R) = 0. \quad (2.25)$$

Boundary conditions are stated at unperturbed surface so as small deviations of size leads to second order magnitude of perturbation. If the local sound velocity at the body surface is not equal zero, Eq. (2.14), (2.25) constitutes that mass density is equal to zero

$$\rho_1(t, r = R) = 0, \quad \text{if} \quad c_s(r = R) \neq 0. \quad (2.26)$$

But if, as it is in polytropic models,  $c_s(r = R) = 0$  then boundary condition (2.25) is valid with arbitrary  $\rho_1(r = R)$ . To find out boundary condition for mass density in this case we integrate (2.8) on the unperturbed volume. Taking into account Gauss theorem we get

$$\frac{d}{dt} \int \rho_1 dV + \int \rho_0 \mathbf{v} d\mathbf{S} = 0, \quad (2.27)$$

where  $d\mathbf{S}$  is body surface element. For zero mass density  $\rho_0$  at  $r = R$  second item in (2.27) is equal to zero and mass conservation law (2.27) reduces to

$$\int \rho_1 dV = 0, \quad \text{if} \quad c_s(r = R) = 0. \quad (2.28)$$

A simplification can now be introduced to handle the non-trivial angular dependence in the perturbation (2.24). It can be rewritten after decomposing variables into spherical harmonics  $Y_{lm}$

$$u(\mathbf{r}) = \sum_{l,m} U_{lm}(r) Y_{lm}(\theta, \phi). \quad (2.29)$$

Usually perturbations with quantum number  $l > 0$  have small increments so we take into account only spherically symmetric perturbations with  $l = 0$ . For this case main equation in spherical system of coordinates is

$$Lu = \frac{\exp(\chi)}{r^2} \frac{d}{dr} \left( r^2 \frac{\exp(-\chi) c_s^2}{\omega^2 + 4\pi G \rho_0} \frac{du}{dr} \right) + u = 0. \quad (2.30)$$

Operator  $L$  has some important features. First of all we now prove that it may have only real eigenvalues  $\omega^2$ . Let multiply (2.30) with complex conjugate  $u^*$  and function  $r^2 \exp(-\chi)$  and integrate along radius. We get

$$- \int_0^R \frac{\exp(-\chi) c_s^2}{\omega^2 + 4\pi G \rho_0} \left| \frac{du}{dr} \right|^2 r^2 dr + \int_0^R \exp(-\chi) |u|^2 r^2 dr = 0. \quad (2.31)$$

First item in (2.31) is derived with integrating by parts and using that  $c_s^2 u$  equals zero on body surface. Subtracting from (2.31) its complex conjugate we get after simple transformation

$$(\omega^2 - \omega^{2*}) \int_0^R \frac{\exp(-\chi) c_s^2}{|\omega^2 + 4\pi G \rho_0|^2} \left| \frac{du}{dr} \right|^2 r^2 dr = 0. \quad (2.32)$$

Obviously integral in equation (2.32) is greater zero and imaginary part of  $\omega^2$  must be zero:  $\text{Im } \omega^2 = 0$ . So there are two types of oscillation modes: if  $\omega^2 > 0$  eigenvalues  $\omega$  are real and introduces sound-like oscillations, but if  $\omega^2 < 0$  eigenvalues are pure imaginary and branch with plus imaginary part introduces monotonously growth of perturbation.

Next, again multiplying equation (2.30) with  $r^2 \exp(-\chi)$  and integrating it along  $r$  we get

$$\left( r^2 \frac{\exp(-\chi) c_s^2}{\omega^2 + 4\pi G \rho_0} \frac{du}{dr} \right) \Big|_0^R + \int_0^R \exp(-\chi) u r^2 dr = 0. \quad (2.33)$$

If  $c_s(R) = 0$  first item in (2.33) equals zero and equation (2.30) automatically conserve total mass (cf. 2.28, 2.23)

$$\int_0^R \exp(-\chi) u r^2 dr = 0. \quad (2.34)$$

To find out conditions for existence of instability suppose that  $\omega^2 < 0$  and mass density  $\rho_0(r)$  and sound velocity  $c_s(r)$  monotonously decrease from their maxima at  $r = 0$  to zero at  $r = R$ . Introduce also function of wave number

$$k(r) = \sqrt{\omega^2 + 4\pi G\rho_0(r)}/c_s(r) = \sqrt{4\pi G\rho_0(r) - |\omega|^2}/c_s(r). \quad (2.35)$$

In this notation equation (2.30) can be rewritten as

$$\frac{\exp(\chi)}{r^2} \frac{d}{dr} \left( r^2 \frac{\exp(-\chi)}{k^2} \frac{du}{dr} \right) + u = 0. \quad (2.36)$$

Wave number  $k(r)$  equals zero at the radius  $r_0$  where

$$4\pi G\rho_0(r_0) - |\omega|^2 = 0. \quad (2.37)$$

At this point equation (2.36) has singularity and eigenfunction  $u$  must have zero derivative

$$\frac{du}{dr}(r = r_0) = 0. \quad (2.38)$$

Now we build up eigenfunctions of (2.36) with method like quasiclassic approach in quantum mechanics ([8]). “Sewing condition” at the point  $r = r_0$  will give us dispersion relation for unstable modes.

Quasiclassic eigenfunction in the region  $r < r_0$  finite at  $r = 0$  is

$$u = \frac{\exp(\chi/2)k^{1/2}}{r} \sin \left( \int_0^r k(r) dr \right) \quad r \ll r_0, \quad k^2 > 0. \quad (2.39)$$

As we can see from (2.39), in this region perturbation oscillate with radius. But in the region  $r > r_0$  it exponentially decrease

$$u = \frac{C \exp(\chi/2)|k|^{1/2}}{2r} \exp \left( - \int_{r_0}^r |k(r)| dr \right) \quad r \gg r_0, \quad k^2 < 0, \quad (2.40)$$

where  $C$  is constant. We will get “sewing condition” from analitical continuation of expression (2.40) into region  $r < r_0$  through up and down halfplanes of the complex variable  $r - r_0$  ([8]). This procedure yields

$$\frac{C \exp(\frac{\chi}{2})|k|^{1/2}}{2r} \exp \left( - \int_{r_0}^r |k(r)| dr \right) \longrightarrow \frac{C \exp(\frac{\chi}{2})k^{1/2}}{r} \sin \left( \int_r^{r_0} k(r) dr + \frac{3\pi}{4} \right). \quad (2.41)$$

Right item of (2.41) must be equal to right item of (2.39)

$$C \sin \left( \int_r^{r_0} k(r) dr + \frac{3\pi}{4} \right) = \sin \left( \int_0^r k(r) dr \right). \quad (2.42)$$

Writing integral at right part of (2.42) in the form

$$\int_0^r k(r) dr = \int_0^{r_0} k(r) dr - \int_r^{r_0} k(r) dr \quad (2.43)$$

we get equality condition

$$\begin{aligned} \int_0^{r_0} k(r) dr &= \frac{\pi}{4} + n\pi \\ n &= 0, 1, 2, \dots \\ C &= (-1)^n. \end{aligned} \tag{2.44}$$

Expressions (2.44) define discrete set of unstable modes with encrements  $\omega_n$  ( $\omega_n^2 < 0$ ) and eigenfunctions of type (2.39), (2.40). Eigenfunction  $u_n$  has strictly  $n$  zero nodes at the interval  $0 < r < R$ . But the mode with  $n = 0$  can not exist as it does not change sign and can not obey the mass conservation law (2.34).

Finally dispersion relation for unstable modes takes the form

$$\int_0^{r_0} \frac{\sqrt{4\pi G \rho_0(r) - |\omega_n|^2}}{c_s(r)} dr = \frac{\pi}{4} + n\pi, \quad n = 1, 2, 3 \dots \tag{2.45}$$

Simple criterion of instability immidiently follows from expression (2.45). If we neglect  $|\omega_n|^2$  and expand integration from  $r_0$  to  $R$  then we obviously get for  $n = 1$  condition for existance of instability in the form

$$\alpha = \frac{4}{5\pi} \int_0^R \sqrt{\frac{4\pi G \rho_0(r)}{c_s^2(r)}} dr > 1. \tag{2.46}$$

### 3. CONCLUSION

Now we can apply criterion (2.46) to models of massive gaseous clouds mainly consisting of molecular hydrogen. At temperatures  $\lesssim 90\text{K}^\circ$  rotational degrees of freedom are degenerated ([7]) and it behaves as monoatomic ideal gas with adiabatic exponent  $\gamma = 5/3$  and polytrope exponent  $n = 3/2$ . At temperatures  $> 90\text{K}^\circ$  adiabatic exponent has standard value  $\gamma = 7/5$  and  $n = 5/2$ . Criterion (2.46) may be rewritten in terms of polytrope model as follows (cf. 2.15, 2.16)

$$\alpha_n = \frac{4}{5\pi} \xi_n \sqrt{n} \int_0^1 \sqrt{\Theta_n^{n-1}(\xi)} d\xi. \tag{3.1}$$

Numeric integration in expression (3.1) gives for  $\gamma = 5/3$

$$\alpha_{3/2} = 0.92, \tag{3.2}$$

and for  $\gamma = 7/5$

$$\alpha_{5/2} = 1.1. \tag{3.3}$$

This result seems to be very intristing. It follows from equations (3.2) and (3.3) that cold hydrogen cloud may be stable relatively gravitational instability, but if the mean

temperature of hydrogen cloud is high enough, it may be the subject of gravitational instability.

We can also remark here that our result for  $\gamma = 5/3$  is applicable to degenerated electron gas (white dwarfs) and our proof of its stability is in accordance with well known results of the theory of white dwarfs ([1],[2],[7]).

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